1. Revisions

Q.1.6.1.1 What is the 2007-th digit after the period in the expansion of $\frac{1}{7}$?
Answer: The decimal expansion of $1/7 = 0.142857142857142857\ldots$ repeats after 6 digits. Since $2007 = 334 \times 6 + 3$, the 2007-th digit is the same as the 3rd digit from the beginning. Thus, 2007-th digit after the period is 2.

Q.1.6.1.2 Express $x = 0.3131313131\ldots$ as a fraction.
Answer: Note that $100x = x + 31$. Thus, $x = \frac{31}{99}$.

Q.1.6.1.3 What is the fraction corresponding to $y = 0.21541541541541541\ldots$?
Answer: Note that $10y = 2 + 0.1541541541541541\cdots = 2 + z$, where $z = 0.154154154\ldots$. Note that $1000z = z + 154 \Rightarrow z = \frac{154}{999}$.
Then, $10y = 2 + \frac{154}{999}$, or, $y = \frac{2152}{9990}$.

Q.1.6.1.4 Prove by contradiction that $\sqrt{2}$ is an irrational number.
Answer: A number is called rational if it can be expressed in the form of $\frac{p}{q}$, where $q \neq 0; p, q$ are integers and p, q are prime to each other.
If possible let $\sqrt{2}$ be a rational number.
So, $\sqrt{2} = \frac{p}{q}$ ($q \neq 0$ and p, q are prime to each other)
$$\text{or, } 2 = \frac{p^2}{q^2}, or, p^2 = 2q^2 \tag{1.1}$$
From (1.1), we see that p^2 contains a factor 2. Since p^2 is a squared number it must have 4 as a factor. So, q must be in the form 2t. Hence p and q have a factor 2 common to them and hence they are not prime to each other. So, our assumption $\sqrt{2}$ is a rational number is false.
So, $\sqrt{2}$ is an irrational number.

Q.1.6.1.5 Identify sets that are not null.
(i) $\{x : x \in \mathbb{Z} \wedge x \in \mathbb{C}\}$, where $\mathbb{Z}$ = the set of integers and $\mathbb{C}$ = the set of complex numbers
(ii) $\{x : x \in \mathbb{Z} \wedge 1 < x < 2\}$
(iii) $\{x : x \in \mathbb{N} - \mathbb{Z}^+\}$, where $\mathbb{Z}^+$ = set of positive numbers and $\mathbb{N}$ = the set of natural numbers
(iv) $\{x : x \in \mathbb{R} \wedge x^2 = -3\}$
Answer: The only set given in (iii) is not null. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$, $\mathcal{Z}^+ =$

$\{1, 2, 3, 4, \ldots\}$. $\therefore \mathbb{N} - \mathbb{Z}^+ = \{0\} \neq \phi$.

Q.1.6.1.6 In the context of three given sets A, B and C, draw a Venn diagram such that (i) $A \cap B \cap C = \phi$, (ii) $A \cap B = \phi$, (iii) $B \cap C \neq \phi$
Answer: Consider that the universal set is being represented by a rectangle in a Venn diagram.

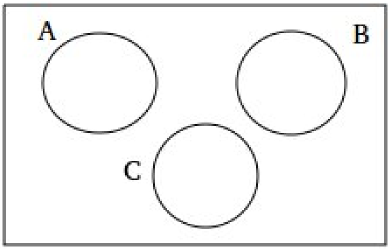

Figure 1.1: Representing the sets when $A \cap B \cap C = \phi$

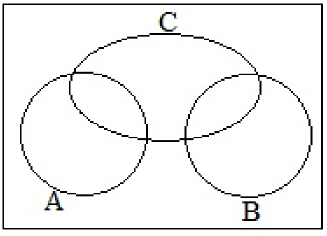

Figure 1.2: Representing the sets when $A \cap B \ = \phi$

Q.1.6.1.7 For any two sets A and B, prove that $A \Delta B = (A \cup B) - (A \cap B)$, where $A \Delta B = \{x : x \in A - B \text{ or } x \in B - A\}$.
Answer: Let $x \in (A \cup B) - (A \cap B) \Leftrightarrow x \in A \cup B$ and $x \notin A \cap B$
$\Leftrightarrow (x \in A \text{ or } x \in B)$ and $x \notin (A \cap B)$
$\Leftrightarrow (x \in A \text{ and } x \notin (A \cap B))$ or $(x \in B \text{ and } x \notin (A \cap B))$
[applying distributive property]
$\Leftrightarrow (x \in A - B)$ or $(x \in B - A)$
$\Leftrightarrow x \in (A - B) \cup (B - A)$

Q.1.6.1.8 A survey on a sample of 25 news paper reading people was

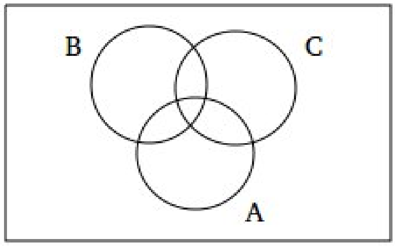

Figure 1.3: Representing the sets when $B \cap C \neq \phi$

conducted. The following facts were found:
16 people read Times of India
12 people read Hindustan Times
11 people read Indian express
8 people read both Times of India and Hindustan Times
8 people read Times of India and Indian Express
3 people read both Hindustan Times and Indian Express
1 people read all the three papers
Find the number of people who read (a) only Indian Express, (b) Times of India but not Hindustan Times, (c) Hindustan Times and Indian Express but not Times of India, (d) only one paper, (e) at least two papers, (f) none of the papers, (g) at least one paper.
Answer: Venn diagram Fig. 1.4 is presented with help of following abbreviations. Symbol 'U' stands for universal set.

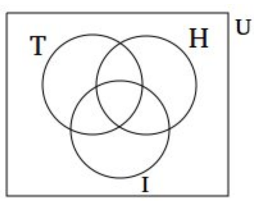

Figure 1.4: Venn diagram for people who read Times of India, Hindustan Times and Indian Express

T: Set of people who read Times of India
H: Set of people who read Hindustan Times
I: Set of people who read Indian Express
$n(T \cup H \cup I) = n(T) + n(H) + n(I) - n(T \cap H) - n(T \cap I) - n(H \cap I) + n(T \cap H \cap I)$

2. Functions and Graphs

Q.1.6.2.1 Find the range of function $f(x) = \frac{1}{x-2}$. Draw the graph.
Answer: Let $y = \frac{1}{x-2}$. Then $x = \frac{1}{y} + 2$.
x can be solved if and only if $y \neq 0$. The range of function f is $\{y \in \mathbb{R} : y \neq 0\}$. Graph of $f(x)$ is shown in Fig. 2.1.

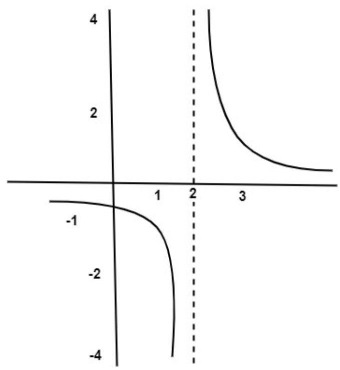

Figure 2.1: Graph of $f(x) = \frac{1}{x-2}$

Q.1.6.2.2 State whether or not each diagram in Fig. 2.2 defines a function from $A = \{x, y, z\}$ into $B = \{1, 2, 3\}$.
Answer: (i) Element y in A is not associated with a unique element in B. So it is not a function.
(ii) $z \in A$ is associated with two elements in B. So, it is not a function.
(iii) It is a function. Each $a \in A$, is associated with a unique element in B.

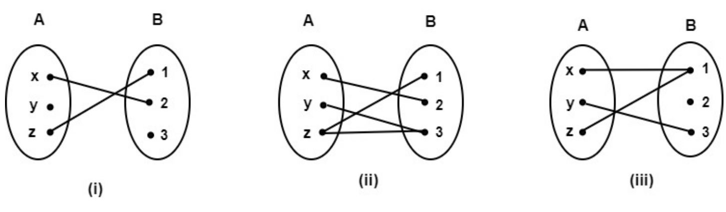

Figure 2.2: Some examples of mappings

Q.1.6.2.3 Let $f = \{(a,\ b),\ (b,\ a),\ (c,\ c)\}$ with domain = co-domain = $\{a,\ b,\ c\}$. Find f^{-1}, f^2.
Answer: $f(a) = b, f(b) = a, f(c) = c$. Therefore , $f^{-1}(b) = a, f^{-1}(a) = b, f^{-1}(c) = c$. Thus , $f^{-1} = \{(b,a),(a,b),(c,c)\}$.
$f^2(a) = f(f(a)) = f(b) = a$. $f^2(b) = f(f(b)) = f(a) = b$. $f^2(c) = f(f(c)) = f(c) = c$. Thus , $f^2 = \{(a,a),(b,b),(c,c)\}$.

Q.1.6.2.4 Find the domain of $f = \sqrt{x+7} - \sqrt{x^2+2x-15}$.
Answer: Function f is defined if and only if $x+7 \geq 0$ and $x^2+2x-15 \geq 0$.
$x + 7 \geq 0 \Rightarrow x \geq -7$ (2.1)
$x^2 + 2x - 15 \geq 0 \Rightarrow x \leq -5$ and $x \geq 3$ (2.2)
Inequalities (2.1) and (2.2) imply that domain of $f = [-7,-5] \cup [3,\infty)$.

Q.1.6.2.5 Prove the following inequality: $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$
Answer: Let $x = n + \epsilon$, where n is a positive integer and $0 \leq \epsilon < 1$.
If $\epsilon = 0$ then $\lfloor x \rfloor = x = \lceil x \rceil = n$.
In this case, $x - 1 = n - 1 < n = x$ and $x + 1 = n + 1 > n = x$.
The above inequality is true for $\epsilon = 0$.
If $0 < \epsilon < 1$ then $\lfloor x \rfloor = n$ and $\lceil x \rceil = n + 1$.
In this case, $\lfloor x \rfloor = n < n + \epsilon = x < n + 1 = \lfloor x \rfloor$
$x - 1 = n - 1 + \epsilon < \lfloor x \rfloor$ and $x + 1 = n + 1 + \epsilon > \lfloor x \rfloor$
The above inequality is true for $0 < \epsilon < 1$. The proof follows in a similar way when n is a negative integer.

Q.1.6.2.6 Draw the graph of $y = x^2 + x$.
Answer: $y = (x + \frac{1}{2})^2 - \frac{1}{4} = (x - (-\frac{1}{2}))^2 + (-\frac{1}{4})$.
$\therefore$ Vertex is $(-\frac{1}{2}, -\frac{1}{4})$, and the line of symmetry is $x = -\frac{1}{2}$.
The graph is shown in Fig. 2.3.

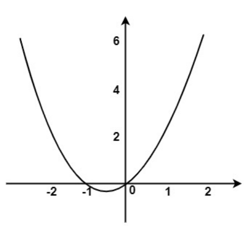

Figure 2.3: Graph of function $y = x^2 + x$

Q.1.6.2.7 Define absolute function, and explain it with the help of graph.
Answer: Absolute function $y = |x|$ is defined as follows.

$$y = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x \leq 0 \end{cases}$$

The absolute function has been given in Fig. 2.4. Both the lines $y = x$ and $y = -x$ make angle 45^0 with the axes.

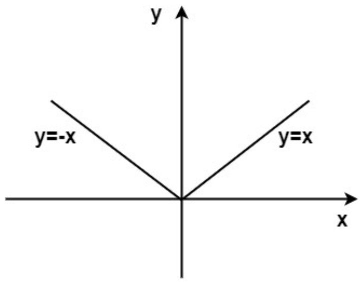

Figure 2.4: Absolute function $y = |x|$

Q.1.6.2.8 Let $f(x) = \lfloor x \rfloor$. Find pre-image set if the range set (i) $\{x\ |0 < x < 1\}$, (ii) $\{-2, -1, 0, 1, 2\}$.
Answer: (i) There is no value in the domain set that correspond to any value in $\{x \mid 0 < x < 1\}$. Pre-image set $= \{\ \} = \phi$.
(ii) Each value in $\{x\ |-2 \leq x < -1\}$ corresponds to -2 in the co-domain.
Each value in $\{x\ |-1 \leq x < 0\}$ corresponds to -1 in the co-domain.
Each value in $\{x\ |0 \leq x < 1\}$ corresponds to 0 in the co-domain.
Each value in $\{x\ |1 \leq x < 2\}$ corresponds to 1 in the co-domain.
Each value in $\{x\ |2 \leq x < 3\}$ corresponds to 2 in the co-domain.
Pre-image set $= \{x\ |-2 \leq x < -1\} \cup \cdots \cup \{x\ |2 \leq x < 3\} = \{x\ |-2 \leq x < 3\}$

Q.1.6.2.9 Function f is defined below. Draw f.

$$f(x) = \begin{cases} x-1, & \text{if } x < -1 \\ -x+1, & \text{if } -1 \leq x \leq 0 \\ x^2, & \text{if } 0 < x \leq 2 \\ x+2, & \text{if } x > 2 \end{cases}$$

Answer: See Fig. 2.5. The function has been defined piecewise. Each piece has a different graph.

Q.1.6.2.10 Let $g(x) = \frac{2x+1}{x^2+1}$. Find the range of g.
Answer: Let $y = \frac{2x+1}{x^2+1}$. We solve for x.
$yx^2 - 2x + (y-1) = 0$

3. Limits

Q.1.6.3.1 If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ then $\lim_{x\to a}[f(x) - g(x)] = 0$. Comment on it.
Answer: The statement is not correct. The limiting values of both f and g are ∞. ∞ concept is something that is unlimited, endless, without bound. Thus, the infinities in both the cases are not necessarily be the same.

Q.1.6.3.2 State and prove leading terms rule.
Answer: Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$, where $a_n \neq 0$ and $b_m \neq 0$. Then $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m}$.
Answer: Now, $\frac{f(x)}{g(x)} = \frac{a_n x^n}{b_m x^m}\phi(x)$, where
$\phi(x) = \frac{1+\frac{a_{n-1}}{a_n}\frac{1}{x}+\cdots+\frac{a_0}{a_n}\frac{1}{x^n}}{1+\frac{b_{m-1}}{b_m}\frac{1}{x}+\cdots+\frac{b_0}{b_m}\frac{1}{x^m}}$. Note that $\lim_{x\to\infty}\phi(x) = 1$.
Thus, the result follows.

Q.1.6.3.3 Find $\lim_{x\to\infty} \frac{sin(x)}{x}$.
Answer: $-1 \leq sin(x) \leq 1$, for any real angle value.
Thus, $\frac{-1}{x} \leq \frac{sin(x)}{x} \leq \frac{1}{x}$
$\lim_{x\to\infty} \frac{-1}{x} = 0$, and also $\lim_{x\to\infty} \frac{1}{x} = 0$.
Hence, $\lim_{x\to\infty} \frac{sin(x)}{x} = 0$. [Squeeze theorem, Q.1.6.3.32]

Q.1.6.3.4 Obtain $\lim_{x\to\infty} \frac{1+x^2}{1+x}$.
Answer: $\lim_{x\to\infty} \frac{1+x^2}{1+x} = \lim_{x\to\infty} \frac{x^2}{x}$ [Leading terms rule, Q.1.6.3.2]
$= \lim_{x\to\infty} x = \infty$. Thus, the limit does not exist.

Q.1.6.3.5 Explain the concept of $\lim_{x\to a} f(x)$.
Answer: Let $f(x)$ be a function defined on an interval that contains $x = a$, except possibly at $x = a$.
We say $\lim_{x\to a} f(x) = k$, if for every number $\epsilon > 0$ there is some number $\delta > 0$ such that $|f(x) - k| < \epsilon$, whenever $0 < |x - a| < \delta$. Fig. 3.1 explains the concept of limit.

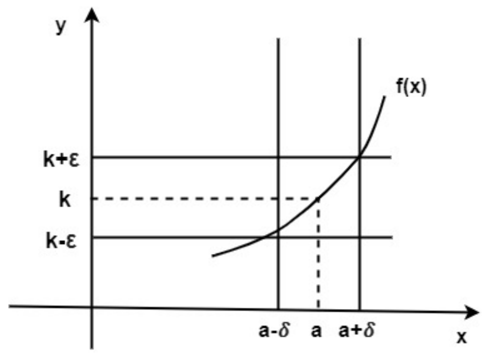

Figure 3.1: Concept of $\lim_{x\to a} f(x) = k$.

Q.1.6.3.6 Applying the definition, show that $\lim_{x\to 6}(-\frac{2}{3}x + 4) = 0$.
Answer: Refer to definition of limit, given in Q.1.6.3.5. We need to find δ using ϵ neighbourhood of 0 in the range of function, and δ depends on ϵ.
Thus, $|(-\frac{2}{3}x + 4) - 0| < \epsilon \Rightarrow |-\frac{2}{3}||x - 6| < \epsilon \Rightarrow |x - 6| < \frac{3}{2}\epsilon$
We take $\delta = \frac{3}{2}\epsilon$.

Q.1.6.3.7 Prove that $\lim_{x\to 2}(x^2 - 3x + 1) = -1$, using the definition of limit.
Answer: Refer to definition of limit, given in Q.1.6.3.5. We need to find δ using the ϵ neighbourhood of -1 in the range of function, and δ depends on ϵ.
Now, $|(x^2 - 3x + 1) - (-1)| < \epsilon \Rightarrow |x^2 - 3x + 2| < \epsilon$
$$\Rightarrow |(x - 1)(x - 2)| < \epsilon \tag{3.1}$$
Let us choose initial vales by restricting δ, and assume that $\delta < 1$.
Then, $|x - 2| < \delta \Rightarrow |x - 2| < 1 \Rightarrow -1 < x - 2 < 1$.
Thus, maximum value of $|x - 1|$ is 2, i.e., $|x - 1| < 2$.
If $2|x - 2| < \epsilon$ then $|(x - 1)(x - 2)| < 2|x - 2| < \epsilon$.
So, $|x - 2| < \frac{\epsilon}{2}$ will lead to $\delta = \frac{\epsilon}{2}$ (3.2)
Combining the assumption $\delta < 1$ and (3.2), let $\delta = minimum\{1, \frac{\epsilon}{2}\}$.

Q.1.6.3.8 Find $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$.
Answer: We need to rationalize the numerator, as it forms $\frac{0}{0}$, after putting 0 for x in the expression.
Answer: $\frac{\sqrt{1+x}-1}{x} = \frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{x(\sqrt{1+x}+1)} = \frac{1+x-1}{x(\sqrt{1+x}+1)} = \frac{x}{x(\sqrt{1+x}+1)}$
$= \frac{1}{\sqrt{1+x}+1}$, since $x \neq 0$

Thus, $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$

Q.1.6.3.9 Does limit exist for $\lim_{x\to 0} \frac{x}{|x|}$?
Answer: Function $|x|$ behaves differently for $x > 0$ and $x < 0$.
$\lim_{x\to 0+} \frac{x}{|x|} = \lim_{x\to 0+} \frac{x}{x} = \lim_{x\to 0+} 1 = 1$
$\lim_{x\to 0-} \frac{x}{|x|} = \lim_{x\to 0+} \frac{x}{-x} = \lim_{x\to 0+} -1 = -1$
Note that $\lim_{x\to 0+} \frac{x}{|x|} \neq \lim_{x\to 0-} \frac{x}{|x|}$. Thus, the limit does not exist.

Q.1.6.3.10 What is $\lim_{x\to a} \frac{x^n-a^n}{x-a}$?
Answer: $x^n - a^n = (x-a)(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$
Thus, $\lim_{x\to a} \frac{x^n-a^n}{x-a} = \lim_{x\to a}(x^{n-1} + x^{n-2}a + \cdots + xa^{n-2} + a^{n-1})$, since $x \neq a$
$= na^{n-1}$.

Q.1.6.3.11 Find $\lim_{h\to 0}(1+h)^{\frac{1}{h}}$.
Answer: Let $n = \frac{1}{h}$. Then $n \to \infty$ as $h \to 0$.
$\lim_{h\to 0}(1+h)^{\frac{1}{h}} = \lim_{n\to\infty}(1+\frac{1}{n})^n$
$= \lim_{n\to\infty}\left[1 + \frac{n}{1!}\frac{1}{n} + \frac{n(n-1)}{2!}\frac{1}{n^2} + \frac{n(n-1)(n-2)}{3!}\frac{1}{n^3} + \dots\right]$
$= \lim_{n\to\infty}\left[1 + \frac{1}{1!} + \frac{1-\frac{1}{n}}{2!} + \frac{(1-\frac{1}{n})(1-\frac{2}{n})}{2!} + \dots\right]$
$= 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e$

Q.1.6.3.12 What is vertical asymptote of a graph? Give an example.
Answer: Let $f(x)$ be a function. If any of the following conditions hold, then the line $x = a$ is a vertical asymptote of $f(x)$.
$\lim_{x\to a-} f(x) = +\infty$, or $-\infty$
$\lim_{x\to a+} f(x) = +\infty$, or $-\infty$
or,
$\lim_{x\to a} f(x) = +\infty$, or $-\infty$
An asymptote is a line that the graph of a function approaches but never touches. It is called vertical, since it is parallel to y-axis, and becomes perpendicular to x-axis. Rational functions contain asymptotes. Consider the function $f(x) = \frac{1}{(x-2)^3}$.
$\lim_{x\to 2-} \frac{1}{(x-2)^3} = -\infty$
$\lim_{x\to 2+} \frac{1}{(x-2)^3} = +\infty$
Thus, $x = 2$ is a vertical asymptote of $f(x)$.

Q.1.6.3.13 Find the vertical asymptotes of the following function: