1. Introduction

Q.1.4.1.1 Show that the number of vertices of odd degree in a graph is always even.
Answer: Let G be a graph having n vertices $v_1, v_2, \ldots, v_n$.
Total degrees of $G = \sum_{i=1}^{n} d(v_i)$, where $d(v_i)$ is the degree of i-th vertex.
Now, total degrees of G =2 $\times$ Total number of edges $(e) = 2e$.

$$\sum_{i=1}^{n} d(v_i) = 2e = \text{an even number} \tag{1.1}$$

Again, $\sum_{i=1}^{n} d(v_i)$
$= \sum_{i \text{ such that } v_i \text{ has even degree}} d(v_i) + \sum_{i \text{ such that } v_i \text{ has odd degree}} d(v_i)$

$$= \text{an even number} + \sum_{i \text{ such that } v_i \text{ has odd degree}} d(v_i) \tag{1.2}$$

By (1.1) and (1.2),
$\sum_{i \text{ such that } v_i \text{ has odd degree}} d(v_i)$
= an even number − another even number = an even number

Q.1.4.1.2 Define k-regular graph using an example.
Answer: A graph is called regular, if each vertex has the same degree. A regular graph with vertices of degree k is called a k-regular graph, or regular graph of degree k. A 3-regular graph is shown in Fig. 1.1.

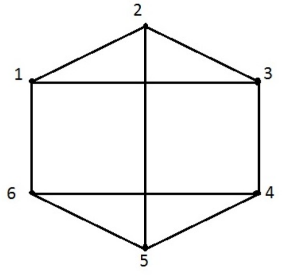


Figure 1.1: A 3-regular graph

In case of a directed graph, $d(v) = d^+(v) + d^-(v)$, where
$d(v)$ = the degree of vertex v
$d^+(v)$ = the out-degree of vertex v
= the number of ougoing edges from vertex v
$d^-(v)$ = the in-degree of vertex v
= the number of incoming edges into vertex v

We discuss more on directed graphs in Chapter 3.

Q.1.4.1.3 Show that there exists a 3-regular graph with n nodes, where n (> 2) is an even number.
Answer: Assume that n is an even number and it is greater than 2. We construct a graph $G = (V, E)$ with n nodes as follows.
Let $V = 0,\ 1,\ 2,\ \ldots,\ n-1$. Construction process of edges in E is given below.
$E = \{(i, i+1) |\ 0 \leq i \leq n-2\} \cup \{(n-1, 0)\} \cup \{(i, i+\frac{n}{2})\ |\ 0 \leq i \leq n/2-1\}$.

Q.1.4.1.4 Explain the notion of isomorphic graphs with the help of an example.
Answer: Two graphs G and G' are said to be isomorphic to each other if there is a one-to-one correspondence (i.e., a bijective mapping) between their sets of vertices
$f : V(G) \to V(G')$
such that any two vertices x and y adjacent in G if and only if $f(x)$ and $f(y$ are adjacent in G'.

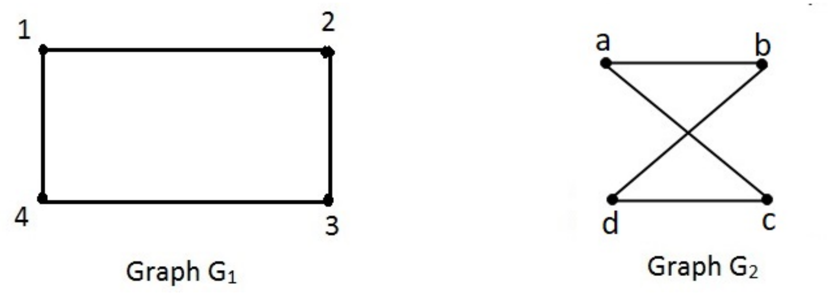


Figure 1.2: Isomorphic graphs G_1 and G_2

Graphs in Fig. 1.2 are isomorphic. An one-to-one mapping is given as follows: $1 \to a, 2 \to b, 3 \to d, 4 \to c$.

Q.1.4.1.5 Are the graphs in Fig. 1.3 isomorphic?
Answer: We establish an one-to-one correspondence between the sets of vertices in graphs i and ii so that the adjacency relationships are preserved.
$v_1 \to e, v_2 \to d, v_3 \to c, v_4 \to g, v_5 \to f, v_6 \to a, v_7 \to b, v_8 \to h$

Thus, there is an one-to-one correspondence (i.e., a bijection) between their vertices. So, the graphs i and ii in Fig. 1.3 are isomorphic.

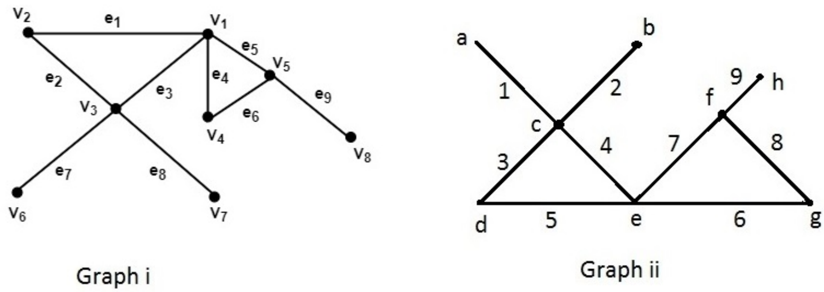


Figure 1.3: Graphs i and ii

Q.1.4.1.6 What is a simple graph?
Answer: We first explain the concepts of a loop and multiple edges. A *loop* is an edge having the same end points. *Multiple edges* are the edges having the same end points.
A graph that does not have a loop or multiple edges is called a simple graph. A simple graph G is shown in Fig. 1.4.

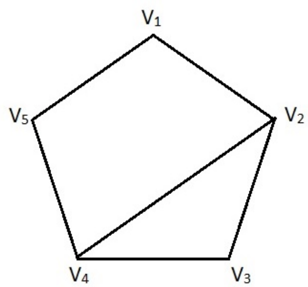


Figure 1.4: A simple graph G

Q.1.4.1.7 Find the number of simple undirected and non-isomorphic graphs with n vertices.
Answer: There are $\binom{n}{2}$ pairs of vertices. For each pair of vertices, either there is an edge, or there is no edge. Thus, we have 2 choices for a pair of vertices. Using product rule, the total number of simple undirected and non-isomorphic graphs is $2 \times 2 \times \cdots \times 2$, repeated for $\binom{n}{2}$ times. Thus, the required number is $2^{\binom{n}{2}}$.

2. Basic Concepts

Q.1.4.2.1 What do you mean by degree of a graph? Give an example.
Answer: The degree of vertex v in a graph G, denoted by $d_G(v)$, is the number of edges that are incident with the vertex v. In a multigraph, loops are counted twice while determining the degree of a vertex. In Fig. 2.1, $d_G(v_2) = 4$ and $d_G(v_4) = 0$.

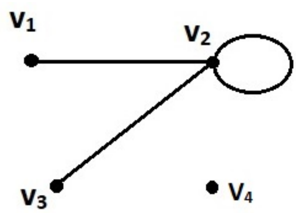


Figure 2.1: Graph G

The degree of a graph is the largest degree among the degrees of the vertces. The degree of the graph G in Fig. 2.1 is 4.

Q.1.4.2.2 State an algorithm for constructing connected simple graph using a degree sequence.
Answer: We state here Havel-Hakimi (HH) algorithm for constructing a connected simple graph using a degree sequence. Imagine that each vertex has as many stubs as its degree. We then need to connect up all these stubs to form a graph. HH algorithm works as follows:
HH algorithm selects an arbitrary vertex, and connect up all of its stubs to the other vertices that have the most free stubs. This is repeated until no unconnected stubs are left.

Q.1.4.2.3 Define the following terms: distance between two vertices and diameter of a graph.
Answer: The distance between two vertices in a graph is the number of edges in a shortest path connecting them.
The diameter of a graph is the largest distance between any pair of vertices.

Q.1.4.2.4 Explain the following terms: empty graph, null graph, singleton graph, trivial graph
Answer: An empty graph with n nodes consists of n isolated nodes, each of 0 edge. It is also called edgeless graph.
Empty graphs with 0 node and 1 node are called the null graph and singleton graph respectively. The null graph is the graph of order 0.
Graph with 1 node is also known as trivial graph. Thus, singleton graph and trivial are the same.

Q.1.4.2.5 Consider the graph given in Fig. 2.2. Find the underlying simple graph.
Answer: The underlying simple graph of G is obtained by deleting all the loops and each collection of parallel edges but one in the collection. The underlying simple graph of G is given below (see Fig. 2.3).

Q.1.4.2.6 Consider the graph in Fig.2.2. Let $V = \{V_1, V_2, V_3\}$. Find the sub-graph $G[V]$ of G induced by V.
Answer: The subgraph $G[V]$ of G induced by V is obtained from the graph G, having vertex set V and edge set consisting of those edges of G that have both ends is V. The induced subgraph $G[V]$ is given in Fig. 2.4.

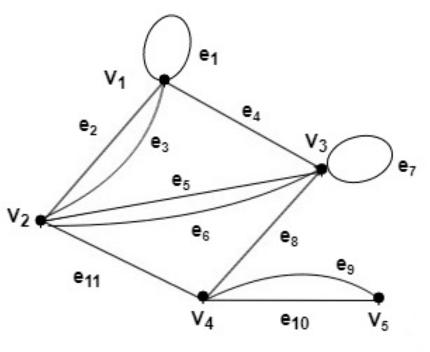


Figure 2.2: Graph G

Q.1.4.2.7 Consider the graph in Fig.2.2. Let $W = \{e_5, e_6, e_7, e_8, e_{11}\}$. Find the sub-graph $G[W]$ of G induced by W.
Answer: The subgraph $G[W]$ of G induced by W is obtained from the

graph G, having edge set W and the associated vertices of edges in W. The induced subgraph $G[W]$ is given below (see Fig. 2.5).

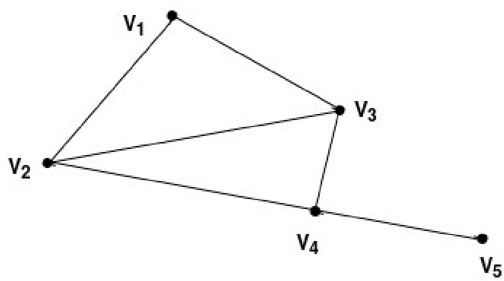


Figure 2.3: The underline simple graph of G given in Fig 2.2

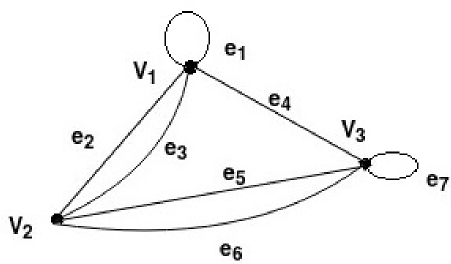


Figure 2.4: Subgraph of G given in Fig 2.2 induced by $\{V_1, V_2, V_3\}$

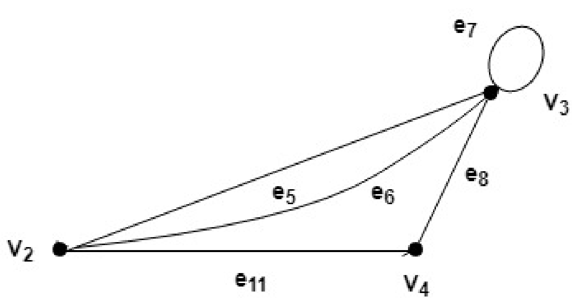


Figure 2.5: Subgraph of G given in Fig 2.2 induced by $\{e_5, e_6, e_7, e_8, e_1 1\}$

3. Directed Graphs

Q.1.4.3.1 Find the complement of the following graph (see Fig. 3.1).

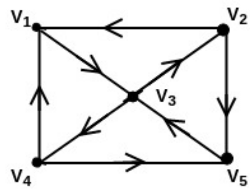


Figure 3.1: Digraph G

Answer: *Defination 3.1*: The complement of a simple digraph $G =< V, E >$ is the digraph $G' =< V, E' >$, where $E' = (V \times V) - E$.
Here, $G =< V, E >$, where $V = \{v_1, v_2, v_3, v_4, v_5\}$
and $E = \{< v_1, v_3 >, < v_2, v_1 >, < v_2, v_5 >, < v_3, v_2 >, < v_3, v_4 >, < v_4, v_1 >, < v_4, v_5 >, < v_5, v_3 >\}$
$E' = (V \times V) - E = \{< v_1, v_1 >, < v_1, v_2 >, < v_1, v_3 >, < v_1, v_4 >, < v_1, v_5 >, < v_2, v_1 >, < v_2, v_2 >, < v_2, v_3 >, < v_2, v_4 >, < v_2, v_5 >, < v_3, v_1 >, < v_3, v_2 >, < v_3, v_3 >, < v_3, v_4 >, < v_3, v_5 >, < v_4, v_1 >, < v_4, v_2 >, < v_4, v_3 >, < v_4, v_4 >, < v_4, v_5 >, < v_5, v_1 >, < v_5, v_2 >, < v_5, v_3 >, < v_5, v_4 >, < v_5, v_5 >\} - E$
$= \{< v_1, v_1 >, < v_1, v_2 >, < v_1, v_4 >, < v_1, v_5 >, < v_2, v_2 >, < v_2, v_3 >, < v_2, v_4 >, < v_3, v_1 >, < v_3, v_3 >, < v_3, v_5 >, < v_4, v_2 >, < v_4, v_3 >, < v_4, v_4 >, < v_5, v_1 >, < v_5, v_2 >, < v_5, v_4 >, < v_5, v_5 >\}$
The diagrammatic representation of $G' = (V, E')$ is given below (see Fig. 3.2):

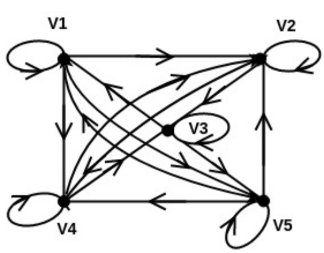


Figure 3.2: Digraph G'

Q.1.4.3.2 Does the following digraph in Fig. 3.3 represent an equivalence relation?

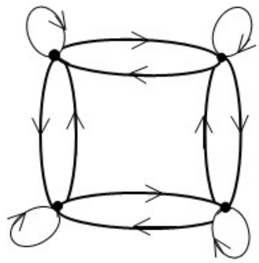


Figure 3.3: A digraph G

Answer: A relation is called an equivalence relation if it is reflexive, symmetric and transitive. Let graph $G =< V, E >$, where $v = \{v_1, v_2, v_3, v_4\}$ and $E = \{< v_1, v_1 >< v_1, v_2 >< v_1, v_4 >, < v_2, v_1 >< v_2, v_2 >< v_2, v_3 >, < v_4, v_4 >\}$
Now $< v_i, v_i > \in E$, for $i = 1, 2, 3, 4$. Therefore, $v_i R v_i$, for every $v_i \in V$. So, relation R is reflexive.
If $< v_i, v_j > \in E$, then $< v_j, v_i > \in E$. If $v_i R v_j$ then $v_j R v_i, \forall i, j = 1, 2, 3, 4$. So, the R is symmetric.
Now $< v_1, v_2 > \in E$ and $< v_2, v_3 > \in E$, but $< v_1, v_3 > \notin E$
Therefore, R is not transitive.
Hence, the digraph in Fig. 3.3 does not represent an equivalence relation.

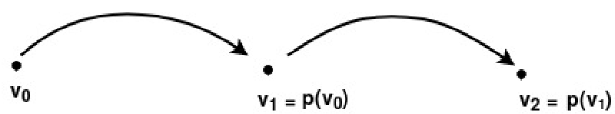
Q.1.4.3.3 Count the number of simple directed graphs with n vertices.
Answer: A simple directed graph does not have self-loops, or multiple edges. For $n = 2$, there are 4 possible simple directed graphs given as follows (see Fig. 3.4):

Figure 3.4: Simple directed graphs i-iv for 2 vertices

There are $\binom{n}{2}$ pairs of vertices. For each pair of vertices there are 4 possible connection choices as seen in Fig. 3.4. Thus, total number of simple directed graphs is $4^{\binom{n}{2}}$.
Note: If we allow no multiple edges but at the most one self loop between a pair of vertices then the number of such digraphs $= 4^{\binom{n}{2}} \times 2^n$, since each vertex has 2 choices, either it has a loop or it does not have a loop.

Q.1.4.3.4 A digraph G has directed edges $< v_1, v_2 >$ and $< v_2, v_1 >$. Then, it is a multi-digraph. Choose the appropriate option.
i. true, ii. false, iii. neither true nor false, iv. both true and false
Answer: ii

Q.1.4.3.5 Every acyclic graph contains at least one node with in-degree 0.
Answer: Let $G =< V, E >$ be a directed graph. We shall prove it by the method of contradiction. Let $d^+(v)$ be the in-degree of vertex v. Assume that $d^+(v) > 0, \forall v \in V$. For each vertex v_i, there is a predecessor $p(v_i)$, such that $< p(v_i), v_i >\in E$.
We start from an arbitrary v_0 to form a list of predecessors as given below. We know that $|V|$ is bounded. Therefore, one must eventually



return to a vertex that was already visited. Hence, there is a cycle. It contradicts the fact that the graph is acyclic.