1. Number Systems

Q.2.1.1.1 Write a note on positional number system.

Answer: Let $\underbrace{d_{n-1}d_{n-2}d_{n-3}\ldots d_0}_{integer\ part} \underbrace{.}_{radix\ point} \underbrace{d_{-1}d_{-2}\ldots d_{-m}}_{fractional\ part}$ be a number in a positional system. Also let r be the base of the number system. It is also called the *radix* of the number system.

This number can be converted into the equivalent decimal number using the following formula.

$$d_{n-1}r^{n-1} + d_{n-2}r^{n-2} + \cdots + d_0r^n + d_{-1}r^{-1} + d_{-2}r^{-2} + \cdots + d_{-m}r^{-m} \quad (1)$$

The number system is positional, since every position has a unique weight. For example, $(n-1)^{th}$ digit d_{n-1} has weight r^{n-1} and m^{th} digit (d_{-m}) after radix point has weight r^{-m}. We require r symbols to represent every number in the system.

Q.2.1.1.2 Convert $(109)_{10}$ into (i) binary, (ii) octal, (iii) hexadecimal numbers, where $(n)_r$ represents number n in system with radix r.

Answer: (i) The equivalent number in binary system is = $(1101101)_2$.

2	109		
2	54	1	
2	27	0	
2	13	1	
2	6	1	
2	3	0	
2	1	1	
	0	1	↑

Symbol ↑ indicates the fact that the remainders are collected from bottom to top.

(ii) The equivalent number in octal system is = $(155)_8$.

8	109		
8	13	5	
8	1	5	
	0	1	↑

(iii) The equivalent number in hexadecimal system is = $(6D)_{16}$.

16	109		
16	6	13	
	0	6	↑

Q.2.1.1.3 Convert $(0.6875)_{10}$ into (i) binary, (ii) octal, (iii) hexadecimal numbers.
Answer: (i) $0.6875 \times 2 = 1.3750 \rightarrow 1$ (whole part)
$0.3750 \times 2 = 0.7500 \rightarrow 0$
$0.7500 \times 2 = 1.5000 \rightarrow 1$
$0.5000 \times 2 = 1.0000 \rightarrow 1$
Hence,
$1 = a_{-1} \downarrow$
$0 = a_{-2}$
$1 = a_{-3}$
$1 = a_{-4}$
$\downarrow$ indicates the fact that the remainders are collected from top to bottom.
Therefore, $(0.6875)_{10} = (0.1011)_2$

(ii) $0.6875 \times 8 = 5.5000 \rightarrow 5$ (whole part)
$0.5000 \times 8 = 4.0000 \rightarrow 4$
Hence,
$5 = a_{-1} \downarrow$
$4 = a_{-2}$
Therefore, $(0.6875)_{10} = (0.54)_8$

(iii) $0.6875 \times 16 = 11.0000 \rightarrow 11$ (whole part)
Hence,
$11 = \mathrm{B} = a_{-1} \downarrow$
Therefore, $(0.6875)_{10} = (0.\mathrm{B})_{16}$

Q.2.1.1.4 Convert the following binary number into octal and hexadecimal numbers using digit-wise conversion: 10110001101011.111100000110
Answer: Each octal digit can be represented by three binary digits, since $2^3 = 8$.
Binary $\rightarrow$ *Octal* conversion:

$$\underbrace{10}_{2}\ \underbrace{110}_{6}\ \underbrace{001}_{1}\ \underbrace{101}_{5}\ \underbrace{011}_{3}\ .\ \underbrace{111}_{7}\ \underbrace{100}_{4}\ \underbrace{000}_{0}\ \underbrace{110}_{6}$$

Therefore, $(10110001101011.111100000110)_2 = (26153.7406)_8$
Again, $2^4 = 16$. Thus, four binary bits are required to represent a hexadecimal digit.
Binary $\rightarrow$ *Hexadecimal* conversion:

$$\underbrace{10}_{2}\,\underbrace{1100}_{C}\,\underbrace{0110}_{6}\,\underbrace{1011}_{B}\,.\,\underbrace{1111}_{F}\,\underbrace{0000}_{0}\,\underbrace{0110}_{6}$$

Therefore, $(10110001101011.111100000110)_2 = (2C6B.F06)_{16}$.

Q.2.1.1.5 Find 2's complement of (i) $(101100)_2$, (ii) $(0.0110)_2$.
Answer: The r's complement of a positive number N in base r having n digits in the integer part is defined as follows:

$$= \begin{cases} r^n - N & N \neq 0 \\ 0, & N = 0 \end{cases}$$

(i) 2's complement of $(101100)_2 = (2^6)_{10} - (101100)_2$
$= (1000000 - 101100)_2 = (010100)_2$
(ii) 2's complement of $(0.0110)_2 = (2^0)_{10} - (0.0110)_2$
$=(1 - 0.0110)_2 = (0.1010)_2$.
The number of digits in the integer part of $(0.0110)_2$ is 0.

Q.2.1.1.6 Obtain 2's complement of $(1101.01)_2$.
Answer: Here, base $r = 2$, and the number of digits in the integer part $n = 4$. Thus, 2's complement of $(1101.01)_2$
$= (2^4)_{10} - (1101.01)_2$, since there are 4 digits in the integer part
$= (10000)_2 - (1101.01)_2 = (10000.00)_2 - (1101.01)_2 = (10.11)_2$

Q.2.1.1.7 Find 9's complement of (i) $(52520)_{10}$, (ii) $(0.3267)_{10}$.
Answer: The $(r-1)$'s complement of a positive number in base r having n digits in the integer part is defined as follows:

$$= \begin{cases} r^n - r^{-m} - N & N \neq 0 \\ 0, & N = 0 \end{cases}$$

where, m is the number of digits in the fractional part of N.
(i) In this case, there is no fractional part. So, $10^{-m} = 10^{-0} = 1$.
Hence, 9's complement of $(52520)_{10} = (10^5 - 1 - 52520) = 99999 - 52520$ $=47479$.
(ii) In this case, there is no integer part. So, $10^n = 10^0 = 1$.
Hence, 9's complement of $(0.3267)_{10} = (1 - 10^{-4} - 0.3267)$
$= 0.9999 - 0.3267 = 0.6732$.

Q.2.1.1.8 Obtain 9's complement of $(6108.02)_{10}$.
Answer: Here, base $r = 10$. The number of digits in the integer part is $n = 4$, and the number of digits in the fractional part is $m = 2$.
Hence, the 9's complement of $(6108.02)_{10}$
$= 10^4 - 10^{-2} - 6108.02 = 10000 - 0.01 - 6108.02 = 3891.97$.

2. Binary Codes

Q.2.1.2.1 Give a few examples of a non-weighted code.
Answer: Non-weighted codes are not positionally weighted. In other words, codes that are not assigned with any weight to each digit position. Excess-3 and Gray codes are non-weighted.

Q.2.1.2.2 Represent number 127 using 8421 BCD and Excess-3 codes.
Answer: The 8421 BCD code is a straight assignment of the binary equivalent. Here weights are assigned to the binary bits according to their positions. Excess-3 code of a digit is obtained by adding 0011 to the 8421 BCD code of the digit. The 8421 BCD and Excess-3 encoding of 127 are 0001 0010 0111 and 0100 0101 1010 respectively.

Q.2.1.2.3 Add 98.90 and 56.392 using 8421 BCD code.
Answer: First we perform addition manually, and then we match the results obtained in both the cases.

```
 98.900
 56.392
 ------
155.292
```

Now we perform addition using 8421 BCD code.

```
  1001   1000   .1001 0000 0000 (98.900 in 8421 BCD)
  0101   0110   .0011 1001 0010 (56.392 in 8421 BCD)
-------------------------------
  1110   1110   .1100 1001 0010 (After adding)
  0110   0110   .0110           (Add 0110 to illegal codes)
-------------------------------
1 0100 1 0100 1 .0010 1001 0010   (Propagate end carry)
1    1      1   .0010 1001 0010   (Add carry)
-------------------------------
1 0101   0101   .0010 1001 0010 (Correct sum - after adding carry)
1  5      5       .2   9    2
```

Q.2.1.2.4 Subtract 98.90 – 56.392 using i. 9's complement method, ii. BCD code.
Answer: First we perform subtraction manually.

```
 98.900
 56.392
 ------
 42.508
```

Now we perform subtraction by 9's complement method.

```
  98.900
  43.607 (9's complement of 56.392)
  ------ (Add)
1 42.507 (There is end carry)
       1
-------- (Add end carry)
  42.508
```

The result matches with the result of manual manual subtraction method. Now we perform subtraction in BCD code.

```
1001 1000 .1001 0000 0000  (98.900 in 8421 BCD)
0101 0110 .0011 1001 0010  (56.392 in 8421 BCD)
-------------------------
0100 0010 .0101 0110 1110  (After subtracting)
                0110 0110  (Subtract 0110, where borrows are present)
-------------------------
0100 0010 .0101 0000 1000  (Correct sum - after subtraction)
  4    2    .5   0    8
```

Q.2.1.2.5 Subtract 98.90 – 56.392 using i. 9's complement method, ii. 9's complement method in 8421 BCD code.
Answer: i. We compute the substraction using 9's complement method.

```
  98.900
  43.607 (9's complement of 56.392)
  ------ (Add)
1 42.507 (There is end carry)
       1
-------- (Add end carry)
  42.508
```

ii. We perform subtraction in 8421 BCD code using 9's complement method.

```
 1001   1000   .1001  0000 0000  (98.900 in 8421 BCD)
 0100   0011   .0110  0000 0111  (43.607 in 8421 BCD)
-------------------------------
 1101   1011   .1111  0000 0111  (After adding)
 0110   0110   .0110             (Add 0110, where code is illegal)
-------------------------------
1 0011 1 0001 1 .0101 0000 0111
     1      1
-------------------------------  (Propagate carry)
1 0100   0010   .0101 0000 0111
                              1  (Add end carry)
-------------------------------
  0100   0010   .0101 0000 1000  (Check the subtraction)
   4      2       .5   0    8
```

Q.2.1.2.6 Subtract 98.90 – 56.392 using i. 10's complement method, ii. 10's complement method in 8421 BCD code.
Answer: i. We compute the substraction using 10's complement method.

```
  98.900
  43.608            (10's complement of 56.392)
  ------            (Add)
1 42.508            (There is end carry)
Answer is = 42.508  (Ignore carry)
```

ii. We perform subtraction in 8421 BCD code using 10's complement method.

```
 1001   1000   .1001  0000 0000  (98.900 in 8421 BCD)
 0100   0011   .0110  0000 1000  (43.608 in 8421 BCD)
-------------------------------
 1101   1011   .1111  0000 1000  (After adding)
 0110   0110   .0110             (Add 0110, where code is illegal)
-------------------------------
1 0011 1 0001 1 .0101 0000 1000
     1      1
-------------------------------  (Propagate carry)
1 0100   0010   .0101 0000 1000
Answer is =  0100   0010   .0101 0000 1000  (Ignore carry)
              4      2       .5   0    8
```

Q.2.1.2.7 Subtract 56.392 – 98.433 using (i) 9's complement method in 8421 BCD code, (ii) 10's complement method in 8421 BCD code.

3. Boolean Algebra

Q.2.1.3.1 What is Boolean algebra?
Answer: Boolean algebra is an algebraic structure defined on a set of elements B, together with two binary operators $+$ and . having the following postulates.
1. Closure properties with respect to operators $+$ and .
2. (i) An identity element with respect to $+$, designated by 0: $x + 0 = 0 + x = x$
(ii) An identity element with respect to . , designated by 1: $x.1 = 1.x = x$
3. Commutative properties (i) $x + y = y + x$, (ii) $x.y = y.x$
4. Distributive properties (i) $x.(y + z) = (x.y) + (x.z)$, (ii) $x + (y.z) = (x + y).(x + z)$
5. For every element $x \in B$, there exists an element $x' \in B$, called the complement of x, such that (i) $x + x' = 1$, and (ii) $x.x' = 0$
6. There exists atleast two elements $x, y \in B$ such that $x \neq y$

Q.2.1.3.2 Define two-valued Boolean algebra.
Answer: A two-valued Boolean algebra is defined on a set of two elements, $B = \{0, 1\}$, with rules for the two binary operators $+$ and . as shown in the following operator tables:

x	y	$x.y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x+y$
0	0	0
0	1	1
1	0	1
1	1	1

x	x'
0	1
1	0

These rules are exactly the same as the AND, OR, and NOT operations, respectively. The postulates stated in Q.2.1.3.1 are valid for the set $B = \{0, 1\}$, and the binary operations as defined in the above tables.

Q.2.1.3.3 Prove the result: $x + x = x$, for a Boolean variable x.
Answer:

$$\begin{aligned} x + x &= (x + x).1 \text{ [2(ii), Q.2.1.3.1]} \\ &= (x + x)(x + x') \text{ [5(i), Q.2.1.3.1]} \\ &= x + xx' \text{ [4(ii), Q.2.1.3.1]} \\ &= x + 0 \text{ [5(ii), Q.2.1.3.1]} \\ &= x \text{ [2(i), Q.2.1.3.1]} \end{aligned}$$

Q.2.1.3.4 Show that $x.0 = 0$.
Answer:

$$\begin{aligned} x.0 &= x.0 + 0 \text{ [2(i), Q.2.1.3.1]} \\ &= x.0 + xx' \text{ [5(ii), Q.2.1.3.1]} \\ &= x(0 + x') \text{ [4(ii), Q.2.1.3.1]} \\ &= xx' \text{ [2(i), Q.2.1.3.1]} \\ &= 0 \text{ [5(ii), Q.2.1.3.1]} \end{aligned}$$

Q.2.1.3.5 Prove the absorption law: $x(x + y) = x$.
Answer:

$$\begin{aligned} x(x+y) &= (x+0)(x+y) \text{ [2(i), Q.2.1.3.1]} \\ &= x + (0.y) \text{ [4(ii), Q.2.1.3.1]} \\ &= x + (y.0) \text{ [3(ii), Q.2.1.3.1]} \\ &= x + 0 \text{ [Q.2.1.3.4]} \\ &= x \text{ [2(ii), Q.2.1.3.1]]} \end{aligned}$$

Q.2.1.3.6 Show that $(x')' = x$.
Answer: We have $x + x' = 1$ and $x.x' = 0$ [5(i), 5(ii), Q.2.1.3.1], where x' is the complement of x. Similarly, complement of x' is x and is also $(x')'$. Since the complement is unique, we have $(x')' = x$.

Q.2.1.3.7 Prove the associative law: $x + (y + z) = (x + y) + z$.
Answer: We shall prove it using truth table method.

xyz	$(x + y)$	$(x + y) + z$	$(y + z)$	$x + (y + z)$
000	0	0	0	0
001	0	1	1	1
010	1	1	1	1
011	1	0	0	0
100	1	1	0	1
101	1	0	1	0
110	0	0	1	0
111	0	1	0	1

Above table shows that the columns corresponding to $x + (y + z)$ and $(x + y) + z$ are the same. This proves the result.

Q.2.1.3.8 Prove that $(x + y)' = x'y'$ [De Morgan's law]
Answer: The law can be proved using truth table method.

x	y	$x+y$	$(x+y)'$	x'	y'	$x'y'$
0	0	0	1	1	1	1
0	1	1	0	1	0	0
1	0	1	0	0	1	0
1	1	1	0	0	0	0

The values under the columns $(x+y)'$ and $x'y'$ are the same for each combination of x and y.

Q.2.1.3.9 There are 2^{2^n} distinct functions for n Boolean variables - Prove it.
Answer: For n Boolean variables, there are 2^n tuples (combinations). For example, there are 2^2 combinations, for 2 variables as given here: {(0,0), (0,1), (1,0), (1,1)}. For each combination, the Boolean function assumes one of possible 2 values. Thus, using multiplication rule, the number of functions is $= 2 \times 2 \times \cdots \times 2$ (2^n times) $= 2^{2^n}$.

Q.2.1.3.10 List all possible functions of two Boolean variables.
Answer: There are $2^{2^n} = 2^{2^2} = 16$ distinct functions. The truth table of all the functions are given below.

x	y	f_0	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	$\ldots$	f_{15}
0	0	0	0	0	0	0	0	0	0	1	1	1	1	$\ldots$	1
0	1	0	0	0	0	1	1	1	1	0	0	0	0	$\ldots$	1
1	0	0	0	1	1	0	0	1	1	0	0	1	1	$\ldots$	1
1	1	0	1	0	1	0	1	0	1	0	1	0	1	$\ldots$	1

From the above table, we get algebraic expression for all the functions. For example, $f_0(x,y) = 0, f_{15}(x,y) = 1, f_1(x,y) = xy, f_2(x,y) = xy'$, $f_3(x,y) = xy' + xy = x$, etc.

Q.2.1.3.11 Simplify the Boolean functions: i. $xy + x'z + yz$, ii. $x + x'y$.
Answer: i. $xy + x'z + yz = xy + x'z + yz(x + x') = xy + x'z + xyz + x'yz$ $= xy(1+z) + x'z(1+y) = xy + x'z$
ii. $x + x'y = (x+x')(x+y) = 1.(x+y) = x+y$

Q.2.1.3.12 Find the complement of the function $f_1(x,y) = x'yz' + x'y'z$.
Answer: $f_1' = (x'yz' + x'y'z)' = (x'yz')'(x'y'z)'$ [De Morgan's law]
$= (x + y' + z)(x + y + z')$ [De Morgan's law]

Q.2.1.3.13 Using the principle of duality, find the complement of