1. Boolean Algebra

Q.1.3.1.1 What is Boolean algebra?
Answer: Boolean algebra is an algebraic structure defined on a set of elements B, together with two binary operators $+$ and $.$ having the following postulates.
1. Closure properties with respect to operators $+$ and $.$
2. (i) An identity element with respect to $+$, designated by 0:
$x + 0 = 0 + x = x$
(ii) An identity element with respect to $.$, designated by 1: $x.1 = 1.x = x$
3. Commutative properties (i) $x + y = y + x$, (ii) $x.y = y.x$
4. Distributive properties (i) $x.(y + z) = (x.y) + (x.z)$,
(ii) $x + (y.z) = (x + y).(x + z)$
5. For every element $x \in B$, there exists an element $x' \in B$, called the *complement* of x, such that (i) $x + x' = 1$, and (ii) $x.x' = 0$
6. There exists atleast two elements $x, y \in B$ such that $x \neq y$

Q.1.3.1.2 Define two-valued Boolean algebra.
Answer: A two-valued Boolean algebra is defined on a set of two elements, $B = \{0, 1\}$, with rules for the two binary operators $+$ and $.$ as shown in the following operator tables (Table 1.1):

Tables 1.1: Operator tables of OR(+), AND(.) and NOT(′)

x	y	$x.y$
0	0	0
0	1	0
1	0	0
1	1	1

x	y	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

x	x'
0	1
1	0

These rules are exactly the same as the AND, OR, and NOT operations, respectively. The postulates stated in Q.Q.1.3.1.1 are valid on the set $B = \{0, 1\}$, and the binary operations as defined in the above tables, see Tables 1.1.

Q.1.3.1.3 Let $< B, +, . >$ be a Boolean algebra. Prove that every element in B has a unique complement.
Answer: Let $a \in B$. If possible, let b and c be two different complements of element a.

$a + b = 1$ [Complement law] (1.1)
$a + c = 1$ [Complement law] (1.2)
$a.b = 0$ [Complement law] (1.3)
$a.c = 0$ [Complement law] (1.4)
Now, $b = b.1$ [Identity law]
$= b.(a + c)$ [From (1.2)]
$= b.a + b.c$ [Distributive law]
$= a.b + b.c$ [Commutative law]
$= 0 + b.c$ [Complement law, (1.3)]
$= b.c$ [Identity law] (1.5)
In a similar manner, $c = c.b = b.c$ [Commutative law] (1.6)
Using (1.5) and (1.6), $b = c$.

Q.1.3.1.4 For every $a \in B$, show that
(i) $a + 1 = 1$
(ii) $a.0 = 0$
Answer: (i) $a + 1 = 1.(a + 1)$ [Identity law]
$= (a + a').(a + 1)$ [Complement law]
$= a + (a'.1)$ [Distributive law]
$= a + a'$ [Identity law]
$= 1$ [Complement law]

(ii) $a.0 = 0 + (a.0)$ [Identity law]
$= (a.a') + (a.0)$ [Complement law]
$= a.(a' + 0)$ [Distributive law]
$= a.a'$ [Identity law]
$= 0$ [Complement law]

Q.1.3.1.5 State and prove absorption laws:
Answer: Absorption laws are stated below.
(a) $x.(x + y) = x$
(b) $x + (x.y) = x$
Answer: (a) $x.(x + y) = (x.x) + (x.y)$ [Distributive law]
$= x + (x.y)$ [see Tables 1.1]
$= (x.1) + (x.y)$ [Identity law]
$= x.(1 + y)$ [Distributive law]
$= x.1$ [Identity law]
$= x$ [Identity law]

(b) $x + (x.y) = (x.1) + (x.y)$ [Identity law]

$= x.(1 + y)$ [Distributive law]
$= x.(y + 1)$ [Commutative law]
$= x.1$ [see Tables 1.1]
$= x$ [Identity law]

Q.1.3.1.6 Prove that $A + A'B = A + B$.
Answer: [Method 1]
RHS $= A + B = A + 1.B$ [Idendity law: $1.x = x$]
$= A + (A + A')B$ [Complement law: $x + x' = 1$]
$= A + AB + A'B$ [Distributive law]
$= A.1 + AB + A'B$ [Identity law]
$= A(1 + B) + A'B$ [Distributive law]
$= A.1 + A'B$ [see Tables 1.1]
$= A + A'B$ [Idendity law: $x.1 = x$]
[Method 2]
LHS $= A + A'B = (A + A')(A + B)$ [Distributive law]
$= 1.(A + B)$ [Complement law: $x + x' = 1$]
$= A + B$ [Identity law]

Q.1.3.1.7 Prove that $AB + A'C + BC = AB + A'C$.
Answer: $AB + A'C + BC$
$= AB + A'C + (A + A')BC$ [Complement law: $x + x' = 1$]
$= AB + A'C + ABC + A'BC$
$= AB + ABC + A'C + A'BC$ [Commutative law]
$= AB(1 + C) + A'C + A'BC$ [Distributive law]
$= AB + A'C + A'BC$ [see Tables 1.1: $1 + x = 1$]
$= AB + A'C + A'CB$ [Commutative law]
$= AB + A'C(1 + B)$ [Distributive law]
$= AB + A'C$ [see Tables 1.1: $1 + x = 1$]

Q.1.3.1.8 Simplify the Boolean expression represented by truth Table 1.2.
Answer: From the truth table, we get
$f(x, y, z) = x'y'z' + x'yz' + xy'z + xyz' + xyz$.
A minterm is added, if f gets 1 for the minterm.
We make algebraic simplification of f as given below.
$f(x, y, z) = x'y'z' + x'yz' + xy'z + xyz' + xyz$
$= x'z'(y' + y) + xy'z + xy(z' + z)$ [Distributive law]
$= x'z' + xy'z + xy$
$= x'z' + x(y'z + y)$ [Distributive law]

2. Order Relations and Lattices

Q.1.3.2.1 What is poset? Give examples.
Answer: Let R be a relation on set S. R is called a partially ordered set (poset) if R is reflexive, antisymmetric, and transitive. Examples of a few posets are given below:
(i) Let N be the set of natural numbers. Consider the relation $\leq$ on N.
$a \leq a$, for $a \in N$. Thus, $\leq$ is reflexive.
If $a \leq b$ and $b \leq a$ then $a = b$. Thus, $\leq$ is antisymmetric.
Also, $a \leq b$ and $b \leq c$ imply that $a \leq c$. So, $\leq$ is transitive.
Thus, $(N, \leq)$ is a poset.
(ii) Another example of poset is a collection of people ordered by genealogical descendancy. Some pairs of people bear the descendant-ancestor relationship, but other pairs of people are incomparable, with neither being a descendant of the other.
(iii) A trivial example of a poset is $(S, =)$, where S is any set. Any set can be partially ordered by equality.

Q.1.3.2.2 Let ' |' be the divisibility operation on $R - \{0\}$. Show that ' |' is not a poset.
Answer: Relation ' |' is not antisymmetric, since $2 \ | -2$ and $-2 \ |2$, but $2 \neq -2$.

Q.1.3.2.3 Define the following terms: total ordering, strict ordering, incomparability, chain.
Answer: Let $(X, \leq)$ be a partial ordered set.
(i) It is total ordered if and only if either $a \leq b$, or $b \leq a$, $\forall a, b \in X$. Total ordering is also called linear ordering.
(ii) It is strictly ordered ($<$) if and only if $a \leq b$ and $a \neq b, a, b \in X$.
(iii) For $a, b \in X$, a and b are incomparable, when neither $a \leq b$, nor $b \leq a$.
(iv) $C \subseteq X$ is a chain if and only if $\leq$ induces a total ordering on C.

Q.1.3.2.4 Identify the relations on $X = \{1, 2, 3, 4\}$, that are not partially otdered.
$R_1 = \{(1,1),(1,2),(3,3),(4,4),(1,3),(3,4),(1,4),(2,2)\}$
$R_2 = \{(1,1),(2,2),(3,1),(1,3),(1,2)\}$
$R_3 = \{(1,1),(2,2),(3,3),(4,4),(1,3),(2,4)\}$

Answer: The binary relations R_1 and R_3 are partial orders. But, R_2 is not a partial order.
$4 \in X$. But $(4,4) \notin R_2$. So, R_2 is not reflexive.
$(3,1),(1,3) \in R_2$. But, $1 \neq 3$. Then R_2 is not antisymmetric.
$(3,1),(1,3) \in R_2$. But, $(3,3) \notin R_2$. Then R_2 is not transitive.
In fact, none of the conditions is satisfied by R_2.

Q.1.3.2.5 What is cover? Give an example.
Answer: Let X be a set with a partial order $\leq$. Let $<$ be the relation on X such that $x < y$ if and only if $x \leq y$ and $x \neq y$.
Let x and y be elements of X. Then y covers x, written as $x \lessdot y$, if $x < y$ and there is no element z such that $x < z < y$. Equivalently, y covers x if the interval $[x, y]$ is the two-element set $\{x, y\}$.
Let $N = \{1, 2, 3, \ldots\}$ be ordered by divisibility (|). Then 15 is covered by 105. But, 14 is not covered by 84, since 14 | 42 | 84.

Q.1.3.2.6 Consider the set $S = \{1, 3, 5, 7, 9\}$. Find all comparable and non-comparable pairs of elements when S is ordered by divisibility.
Answer: Comparable pairs on S are (1, 3), (1, 5), (1, 7), (1, 9), (3, 9).
Non-comparable pairs are (3, 5), (3, 7), (5, 7), (5, 9), (7, 9).
Note that elements x and y are non-comparable if neither $x \mid y$ nor $y \mid x$.

Q.1.3.2.7 Let $X = (1, 2, 3, \ldots)$ be ordered by divisibility. Check whether the subset (2, 4, 8, 12) is linearly ordered.
Answer: A set S is said to linearly ordered if every pair of elements is comparable. A linearly ordered set is also called totally ordered. Here the pair (8, 12) is not comparable, since neither 8 divides 12 nor 12 divides 8. Thus, the given set X is not linearly ordered.

Q.1.3.2.8 Let N be the set of natural number ordered by $\leq$. Thus, $(a, b) \leq (a', b')$ if $a \leq a'$ and $b \leq b'$. Let us define an order $<$ on $N \times N$ such that $(a, b) < (a', b')$, if $(a \leq a'$ and $b < b')$, or $(a < a'$ and $b \leq b')$.
Order the pairs (i) (1, 4) and (1, 3), (ii) (1, 4) and (1, 5).
Answer: (i) $(1,4) > (1,3)$, since $(1 \geq 1)$ and $(4 > 3)$
(ii) $(1,4) < (1,5)$, since $(1 \leq 1)$ and $(4 < 5)$

Q.1.3.2.9 What is anti-chain? Give examples.
Answer: Let $(X, \leq)$ be a partial ordered set. $D \subseteq X$ is an anti-chain if every pair of elements in D are incomparable.
The following sets are antichains based on of subset relationship:

(i) $\{\{1\},\{2\},\{3\}\}$
(ii) $\{\{1,2\},\{2,3\},\{1,3\}\}$

Q.1.3.2.10 What is product partial order?
Answer: A partial order $\leq$ defined on the cartesian product $S \times T$ is known as product partial order, where $(S, \leq)$ and $(T, \leq)$ are posets.

Q.1.3.2.11 Explain the concept of Hasse diagram with help of an example.
Answer: A Hasse diagram is a graphical representation of a partially ordered set. The graph is displayed using the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the poset, and line segments are drawn between these points according to the following rules:
(i) If $x < y$ in the poset, then the point corresponding to x appears lower in the drawing than the point corresponding to y.
(ii) A line segment between the points corresponding to any two elements x and y of the poset is included in the drawing if and only if x covers y, or y covers x.
Consider the set $X = \{1,2,3\}$. Let the power set of X be $\rho(X)$. Then $(\rho(X), \subseteq)$ is a poset as depicted using Hasse diagram in Fig. 2.1.

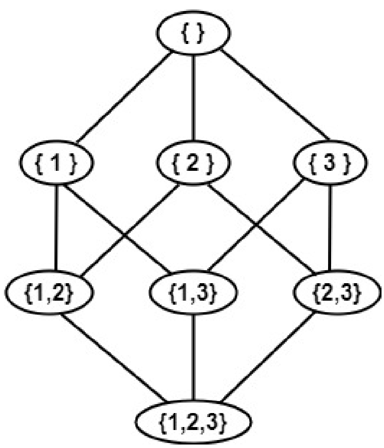

Figure 2.1: Hasse diagram of $(\rho(X), \subseteq)$

Here, $\rho(X) = \{\phi, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.

Q.1.3.2.12 Draw Hasse diagram of poset containing factors of 60, partially ordered by divisibility.

3. Fibonacci Numbers

Q.1.3.3.1 Define Fibonacci series. Find 10-th Fibonacci number.
Answer: The Fibonacci series has been defined by the following recurrence relation: $F_{n+1} = F_n + F_{n-1}$ with base conditions: $F_1 = 1, F_2 = 1$. Based on the above recurrence relation and base conditions, we compute the following numbers.
$F_3 = F_2 + F_1 = 1 + 1 = 2$
$F_4 = F_3 + F_2 = 2 + 1 = 3$
$F_5 = F_4 + F_3 = 3 + 2 = 5$
$F_6 = F_5 + F_4 = 5 + 3 = 8$
$F_7 = F_6 + F_5 = 8 + 5 = 13$
$F_8 = F_7 + F_6 = 13 + 8 = 21$
$F_9 = F_8 + F_7 = 21 + 12 = 34$
$F_{10} = F_9 + F_8 = 34 + 21 = 55$

Q.1.3.3.2 The Fibonacci numbers can be extended to zero and negative indices using the relation $F_n = F_{n+2} - F_{n+1}$. Calculate Fibonacci numbers from F_0 to F_{-5}. Find a general formula for F_{-n} in terms of F_n. Prove your result.
Answer: Fibonacci numbers from F_0 to F_{-5} are given below:
$F_0 = F_2 - F_1 = 0$
$F_{-1} = F_1 - F_0 = 1 - 0 = 1$
$F_{-2} = F_0 - F_1 = 0 - 1 = -1$
$F_{-3} = F_{-1} - F_{-2} = 1 - (-1) = 2$
$F_{-4} = F_{-2} - F_{-3} = -1 - 2 = -3$
$F_{-5} = F_{-3} - F_{-4} = 2 - (-3) = 5$
It can be shown that $F_{-n} = (-1)^{n+1}F_n$. We shall prove the formula using mathematical induction on n.
Based on the above calculations, $F_{-1} = 1 = (-1)^{1+1}F_1$.
$F_{-2} = -1 = (-1)^{2+1}F_2$ [Q.1.3.3.1]
So, the formula holds true for $n = 1$ and $n = 2$.
We assume that the formula is true for $n \leq k$. We shall prove that the formula is true for $n = k + 1$.
Now, $F_{-(k+1)} = F_{-(k+1)+2} - F_{-(k+1)+1}$ [From definition]
$= F_{-(k-1)} - F_{-k} = (-1)^k F_{k-1} - (-1)^{k+1} F_k$ [Induction hypothesis]
$= (-1)^{k+2}(F_{k-1} + F_k)$
$= (-1)^{(k+1)+1}F_{k+1}$ [From recurrence relation]

Q.1.3.3.3 Obtain the generating function of Fibonacci numbers.
Answer: The generating function is given by $f(x) = \sum_{n=1}^{\infty} F_n x^n$.
Then $f(x) = F_1x + F_2x^2 + F_3x^3 + F_4x^4 + \ldots$
$xf(x) = F_1x^2 + F_2x^3 + F_3x^4 + F_4x^5 + \ldots$
$x^2f(x) = F_1x^3 + F_2x^4 + F_3x^5 + F_4x^6 + \ldots$
Now, $(1 - x - x^2)f(x) = f(x) - xf(x) - x^2f(x)$
We apply the facts that $F_1 = F_2 = 1$ and $F_{n+1} - F_n - F_{n-1} = 0$.
Then $f(x) - xf(x) - x^2f(x) = x$. Thus, $(1 - x - x^2)f(x) = x$
or, $f(x) = \frac{x}{1-x-x^2}$.

Q.1.3.3.4 State some properties of Fibonacci numbers.
Answer: Some interesting properties of Fibonacci numbers are given below:
Property 1: $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ (3.1)
Proof: $F_1 = F_3 - F_2$
$F_2 = F_4 - F_3$
$F_3 = F_5 - F_4$
$\ldots$
$F_{n-1} = F_{n+1} - F_n$
$F_n = F_{n+2} - F_{n+1}$
By adding, $\sum_{i=1}^{n} F_i = F_{n+2} - F_2 = F_{n+2} - 1$, since $F_2 = 1$.

Property 2: $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ (3.2)
Proof: $F_1 = F_2$
$F_3 = F_4 - F_2$
$F_5 = F_6 - F_4$
$\ldots$
$F_{2n-1} = F_{2n} - F_{2n-2}$
By adding, we gat $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$

Property 3: $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ (3.3)
Proof: From(3.1), $\sum_{i=1}^{2n} F_i = F_{2n+2} - 1$ (3.4)
(3.4) – (3.2) $\Rightarrow$
$\sum_{i=1}^{n} F_{2i} = F_{2n+2} - 1 - F_{2n}$
$= F_{2n+2} - F_{2n} - 1$
$= F_{2n+1} - 1$

Corollary: $F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-1} - F_{2n}$
$= -F_{2n-1} + 1$ (3.5)

Proof: It follows from (3.2) and (3.3).
Corollary: $F_1 - F_2 + F_3 - F_4 + \cdots - F_{2n} + F_{2n+1} = F_{2n} + 1$ (3.6)
Proof: By adding F_{2n+1} to both sides of (3.5), the result follows.
Corollary: $F_1 - F_2 + F_3 - F_4 + \ldots (-1)^{n+1} F_n$
$= (-1)^{n+1} F_{n-1} + 1$ (3.7)
Proof: Combining (3.5) and (3.6), the result follows.

Property 4: $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$ (3.8)
Proof: We note that $F_k F_{k+1} - F_{k-1} F_k = F_k(F_{k+1} - F_{k-1}) = F_k^2$
$F_1^2 = F_1 F_2$ (as $F_1 = F_2 = 1$)
$F_2^2 = F_2 F_3 - F_1 F_2$
$F_3^2 = F_3 F_4 - F_2 F_3$
...
$F_n^2 = F_n F_{n+1} - F_{n-1} F_n$
By adding, we get $\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$

Q.1.3.3.5 Solve the following recurrence relation.
$u_n = u_{n-1} + u_{n-2}$ (3.9)
where, $u_1 = u_2 = 1$
Answer: Using (3.9), $\sum_{n=3}^{\infty} u_n x^h = \sum_{n=3}^{\infty} u_{n-1} x^n + \sum_{n=3}^{\infty} u_{n-2} x^n$
Let $G(x) = u_1 + u_2 x + u_3 x^2 + \cdots + u_{n-1} x^{n-2} + u_n x^{n-1} + \ldots$
Therefore, $x \sum_{n=3}^{\infty} u_n x^{n-1} = x^2 \sum_{n=3}^{\infty} a_{n-1} x^{n-2} + x^3 \sum_{n=3}^{\infty} a_{n-2} x^{n-3}$
or, $G(x) - u_1 - u_2 x = x[G(x) - u_1] + x^2 G(x)$, for $x \neq 0$
or, $G(x)[1 - x - x^2] = u_1 + u_2 x - u_1 x = u_1 = 1$ as $u_2 = u_2 = 1$
or, $G(x) = (1 - x - x^2)^{-1} = \frac{1}{1-x-x^2}$
or, $G(x) = \frac{1}{(x-\alpha)(x-\beta)}$, where α, β are the roots of equation $1 - x - x^2 = 0$
or $x^2 + x - 1 = 0$
i.e. $\alpha = \frac{-1+\sqrt{5}}{2}, \beta = \frac{-1-\sqrt{5}}{2}$
or, $G(x) = -\frac{1}{\alpha-\beta}[\frac{1}{x-\alpha} - \frac{1}{x-\beta}]$
or, $G(x) = -\frac{1}{\sqrt{5}}[\frac{1}{x-\alpha} - \frac{1}{x-\beta}]$
or, $G(x) = +\frac{1}{\sqrt{5}}[\frac{1}{\alpha}(1 - \frac{x}{\alpha})^{-1} - \frac{1}{\beta}(1 - \frac{x}{\beta})^{-1}]$ (3.10)
(3.10)is an identity.
Then, $u_n =$ co-efficient of x^{n-1}in the right side of (3.10)
$u_n = \frac{1}{\sqrt{5}}[\frac{1}{\alpha}.\frac{1}{\alpha^{n-1}} - \frac{1}{\beta}.\frac{1}{\beta^{n-1}}] = \frac{1}{\sqrt{5}}[\frac{1}{\alpha^n} - \frac{1}{\beta^n}]$
So, $u_n = \frac{1}{\sqrt{5}}[\frac{\beta^n - \alpha^n}{(\alpha\beta)^n}] = \frac{\alpha^n - \beta^n}{\sqrt{5}}$,
where $\alpha\beta = \frac{1}{4}((-1)^2 - (\sqrt{5})^2) = -1$
Thus, $u_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}}$ (3.11)
Formula (3.11) is called Binet's formula in honour of the mathematician