# 1. Boolean algebra

Q.1.1 What is Boolean algebra?
Answer: Boolean algebra is an algebraic structure defined on a set of elements $B$, together with two binary operators $+$ and . having the following postulates.
1. Closure properties with respect to operators $+$ and .
2. (i) An identity element with respect to $+$, designated by 0:
$x + 0 = 0 + x = x$
(ii) An identity element with respect to . , designated by 1: $x.1 = 1.x = x$
3. Commutative properties (i) $x + y = y + x$, (ii) $x.y = y.x$
4. Distributive properties (i) $x.(y + z) = (x.y) + (x.z)$,
(ii) $x + (y.z) = (x + y).(x + z)$
5. For every element $x \in B$, there exists an element $x' \in B$, called the *complement* of $x$, such that (i) $x + x' = 1$, and (ii) $x.x' = 0$
6. There exists atleast two elements $x, y \in B$ such that $x \neq y$

Q.1.2 Define two-valued Boolean algebra.
Answer: A two-valued Boolean algebra is defined on a set of two elements, $B = \{0, 1\}$, with rules for the two binary operators $+$ and . as shown in the following operator tables (Table 1.1):

**Tables 1.1:** Operator tables of OR($+$), AND(.) and NOT($'$)

| $x$ | $y$ | $x.y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

| $x$ | $y$ | $x + y$ |
|---|---|---|
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

| $x$ | $x'$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

These rules are exactly the same as the AND, OR, and NOT operations, respectively. The postulates stated in Q.1.1 are valid on the set $B$ = $\{0, 1\}$, and the binary operations as defined in Tables 1.1.

Q.1.3 Let $< B, +, . >$ be a Boolean algebra. Prove that every element in $B$ has a unique complement.
Answer: Let $a \in B$. If possible, let $b$ and $c$ be two different complements of element $a$.
$a + b = 1$ [Complement law] $\hspace{3cm}$ (1.1)

$a + c = 1$ [Complement law] $\hspace{4cm}$ (1.2)

$a.b = 0$ [Complement law] $\hspace{4cm}$ (1.3)

$a.c = 0$ [Complement law] $\hspace{4cm}$ (1.4)

Now, $b = b.1$ [Identity law]

$= b.(a + c)$ [From (1.2)]

$= b.a + b.c$ [Distributive law]

$= a.b + b.c$ [Commutative law]

$= 0 + b.c$ [Complement law, (1.3)]

$= b.c$ [Identity law] $\hspace{4cm}$ (1.5)

In a similar manner, $c = c.b = b.c$ [Commutative law] $\hspace{1.5cm}$ (1.6)

Using (1.5) and (1.6), $b = c$.

Q.1.4 For every $a \in B$, show that

(i) $a + 1 = 1$

(ii) $a.0 = 0$

Answer: (i) $a + 1 = 1.(a + 1)$ [Identity law]

$= (a + a').(a + 1)$ [Complement law]

$= a + (a'.1)$ [Distributive law]

$= a + a'$ [Identity law]

$= 1$ [Complement law]

(ii) $a.0 = 0 + (a.0)$ [Identity law]

$= (a.a') + (a.0)$ [Complement law]

$= a.(a' + 0)$ [Distributive law]

$= a.a'$ [Identity law]

$= 0$ [Complement law]

Q.1.5 State and prove absorption laws:

Answer: Absorption laws are stated below.

(a) $x.(x + y) = x$

(b) $x + (x.y) = x$

Answer: (a) $x.(x + y) = (x.x) + (x.y)$ [Distributive law]

$= x + (x.y)$ [see Tables 1.1]

$= (x.1) + (x.y)$ [Identity law]

$= x.(1 + y)$ [Distributive law]

$= x.1$ [Identity law]

$= x$ [Identity law]

(b) $x + (x.y) = (x.1) + (x.y)$ [Identity law]

$= x.(1 + y)$ [Distributive law]

$= x.(y + 1)$ [Commutative law]
$= x.1$ [Tables 1.1]
$= x$ [Identity law]

Q.1.6 Prove that $A + A'B = A + B$.
Answer: [Method 1]
RHS $= A + B = A + 1.B$ [Idendity law: $1.x = x$]
$= A + (A + A')B$ [Complement law: $x + x' = 1$]
$= A + AB + A'B$ [Distributive law]
$= A.1 + AB + A'B$ [Identity law]
$= A(1 + B) + A'B$ [Distributive law]
$= A.1 + A'B$ [Tables 1.1]
$= A + A'B$ [Idendity law: $x.1 = x$]
[Method 2]
LHS $= A + A'B = (A + A')(A + B)$ [Distributive law]
$= 1.(A + B)$ [Complement law: $x + x' = 1$]
$= A + B$ [Identity law]

Q.1.7 Prove that $AB + A'C + BC = AB + A'C$.
Answer: $AB + A'C + BC$
$= AB + A'C + (A + A')BC$ [Complement law: $x + x' = 1$]
$= AB + A'C + ABC + A'BC$ [Distributive law]
$= AB + ABC + A'C + A'BC$ [Commutative law]
$= AB(1 + C) + A'C + A'BC$ [Distributive law]
$= AB + A'C + A'BC$ [Tables 1.1: $1 + x = 1$]
$= AB + A'C + A'CB$ [Commutative law]
$= AB + A'C(1 + B)$ [Distributive law]
$= AB + A'C$ [Tables 1.1: $1 + x = 1$]

Q.1.8 Simplify the Boolean expression represented by truth Table 1.2.
Answer: From the truth table, we get
$f(x, y, z) = x'y'z' + x'yz' + xy'z + xyz' + xyz$.
A minterm is added, if $f$ gets 1 for the minterm.
We make algebraic simplification of $f$ as given below.
$f(x, y, z) = x'y'z' + x'yz' + xy'z + xyz' + xyz$
$= x'z'(y' + y) + xy'z + xy(z' + z)$ [Distributive law]
$= x'z' + xy'z + xy$
$= x'z' + x(y'z + y)$ [Distributive law]
$= x'z' + x(y' + y)(z + y)$ [Distributive law]
$= x'z' + x(z + y)$ [Complement law: $y + y' = 1$]

# 2. Order relations and lattices

Q.2.1 What is poset? Give a few examples.

Answer: Let $R$ be a relation on set $S$. $R$ is called a partially ordered set (poset) if $R$ is reflexive, antisymmetric, and transitive. Examples of a few posets are given below:

(i) Let $\mathbb{N}$ be the set of natural numbers. Consider the relation $\leq$ on $\mathbb{N}$. It is trivial that $a \leq a$, for $a \in \mathbb{N}$. Thus, $\leq$ is reflexive.

If $a \leq b$ and $b \leq a$ then $a = b$. Thus, $\leq$ is antisymmetric.

Also, $a \leq b$ and $b \leq c$ imply that $a \leq c$. So, $\leq$ is transitive.

Thus, $(\mathbb{N}, \leq)$ is a poset.

(ii) Let $\mathbb{Z}^+$ be the set of positive numbers. We shall show that $(\mathbb{Z}^+, |)$ is a poset.

It is trivial that $a|a$, for $a \in \mathbb{Z}^+$. Thus, $|$ is reflexive.

If $a|b$ and $b|a$ then $a = b$. Thus, $|$ is antisymmetric.

Also, $a|b$ and $b|c$ imply that $a|c$. So, $|$ is transitive.

(iii) A trivial example of a poset is $(S, =)$, where $S$ is any set. Any set can be partially ordered by equality.

Q.2.2 Let ' $|$ ' be the divisibility operation on $R - \{0\}$. Show that ' $|$ ' is not a poset.

Answer: Relation ' $|$ ' is not antisymmetric, since $2 \mid -2$ and $-2 \mid 2$, but $2 \neq -2$.

Q.2.3 Define the following terms: total ordering, strict ordering, incomparability, chain.

Answer: Let $(X, \leq)$ be a partial ordered set.

It is total ordered (comparable) if and only if either $a \leq b$, or $b \leq a$, $\forall a, b \in X$. If neither $a \leq b$ nor $b \leq a$, the elements $a$ and $b$ are incomparable. Total ordering is also called linear ordering. For example, let $S = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$. Note that elements $\{1\}$ and $\{2\}$ of $S$ are not comparable. Then, $(S, \subseteq)$ is not totally ordered. But, $(S, \subseteq)$ is a poset (check!). Thus, set $S$ is not totally ordered under $\subseteq$ operation.

It is strictly ordered $(<)$ if and only if $a \leq b$ and $a \neq b, a, b \in X$. Note that elements $\{1\}$ and $\{1, 2\}$ of $S$ are such that $\{1\} \subseteq \{1, 2\}$, but, $\{1\} \neq \{1, 2\}$. So, elements $\{1\}$ and $\{1, 2\}$ of $S$ are strict comparable under operation $\subseteq$. [Q.2.3(i)]

$C \subseteq X$ is a chain if and only if $\leq$ induces a total ordering on $C$. Note

that elements $\{1\}$ and $\{1,2\}$ of $S$ are comparable under operation $\subseteq$, since $\{1\} \subseteq \{1,2\}$. So, $Y = \{\{1\},\{1,2\}\}$ is a chain, where $Y \subseteq S$. [Q.2.3(i)]

Q.2.4 Identify the following relations on $X = \{1,2,3,4\}$, that are not partially ordered.
$R_1 = \{(1,1),(1,2),(3,3),(4,4),(1,3),(3,4),(1,4),(2,2)\}$
$R_2 = \{(1,1),(2,2),(3,1),(1,3),(1,2)\}$
$R_3 = \{(1,1),(2,2),(3,3),(4,4),(1,3),(2,4)\}$
Answer: The binary relations $R_1$ and $R_3$ are partial orders. But, $R_2$ is not a partial order.
First, we shall show that $R_2$ is not a partial ordered set.
$4 \in X$. But $(4,4) \notin R_2$. So, $R_2$ is not reflexive.
$(3,1),(1,3) \in R_2$. But, $1 \neq 3$. Then $R_2$ is not antisymmetric.
$(3,1),(1,3) \in R_2$. But, $(3,3) \notin R_2$. Then $R_2$ is not transitive.
In fact, none of the conditions is satisfied by $R_2$.
Now, we shall show that $R_1$ is not a partial ordered set.
For each $x \in X, (x,x) \in R_1$. So, $R_1$ is reflexive.
$(1,3),(3,4) \in R_2$. Also, $(1,4) \in R_2$. So, $R_1$ is transitive.
$R_1$ is not antisymmetric, since no such pair inclusion / exclusion goes against it.
Thus, relation $R_1$ is a partial ordered set.
In the same way, check that $R_3$ is a partial ordered set.

Q.2.5 What is cover? Give an example.
Answer: Let $X$ be a set with a partial order $\leq$. Let $<$ be the relation on $X$ such that $x < y$ if and only if $x \leq y$ and $x \neq y$.
Let $x$ and $y$ be elements of $X$. Then $y$ covers $x$, written as $x \lessdot y$, if $x < y$ and there is no element $z$ such that $x < z < y$. Equivalently, $y$ covers $x$ if the interval $[x,y]$ is the two-element set $\{x,y\}$.
Let $N = \{1,2,3,\dots\}$ be ordered by divisibility ( $\mid$ ). Then 15 is covered by 105. But, 14 is not covered by 84, since $14 \mid 42 \mid 84$.

Q.2.6 Consider the set $S = \{1,3,5,7,9\}$. Find all comparable and non-comparable pairs of elements, when $S$ is ordered by divisibility.
Answer: Note that elements $x$ and $y$ are non-comparable if neither $x \mid y$, nor $y \mid x$.
Comparable pairs on $S$ are (1, 3), (1, 5), (1, 7), (1, 9), (3, 9).
Non-comparable pairs are (3, 5), (3, 7), (5, 7), (5, 9), (7, 9).

Q.2.7 Let $X = (1, 2, 3, \dots)$ be ordered by divisibility. Check whether the subset (2, 4, 8, 12) is linearly ordered.
Answer: A set $S$ is said to linearly ordered if every pair of elements is comparable. A linearly ordered set is also called totally ordered. Here, the pair (8, 12) is not comparable, since neither 8 divides 12 nor 12 divides 8. Thus, the given set $X$ is not linearly ordered.

Q.2.8 Let $N$ be the set of natural number ordered by $\leq$. Thus, $(a, b) \leq (a', b')$, if $a \leq a'$ and $b \leq b'$. Let us define an order $<$ on $N \times N$ such that $(a, b) < (a', b')$, if $(a \leq a'$ and $b < b')$, or $(a < a'$ and $b \leq b')$.
Order the pairs (i) (1, 4) and (1, 3), (ii) (1, 4) and (1, 5).
Answer: (i) $(1, 4) > (1, 3)$, since $(1 \geq 1)$ and $(4 > 3)$
(ii) $(1, 4) < (1, 5)$, since $(1 \leq 1)$ and $(4 < 5)$

Q.2.9 What is anti-chain? Give examples.
Answer: Let $(X, \leq)$ be a partial ordered set, where, $X = \{1, 2, 3, \dots\}$. $D \subseteq X$ is an anti-chain, if every pair of elements in $D$ are incomparable. The following sets are anti-chains based on of subset relationship:
(i) $\{\{1\}, \{2\}, \{3\}\}$
(ii) $\{\{1, 2\}, \{2, 3\}, \{1, 3\}\}$

Q.2.10 What is product partial order?
Answer: A partial order $\leq$ defined on the cartesian product $S \times T$ is known as product partial order, where $(S, \leq)$ and $(T, \leq)$ are posets.
Let $(s_1, t_1), (s_2, t_2) \in S \times T$. Then $(s_1, t_1) \leq (s_2, t_2)$ if $s_1 \leq s_2$ in $S$ and $t_1 \leq t_2$ in $T$.

Q.2.11 Explain the concept of Hasse diagram with help of an example.
Answer: A Hasse diagram is a graphical representation of a partially ordered set. The graph is displayed using the cover relation of the partially ordered set with an implied upward orientation. A point is drawn for each element of the poset, and line segments are drawn between these points according to the following rules:
(i) If $x < y$ in the poset, then the point corresponding to $x$ appears lower in the drawing than the point corresponding to $y$.
(ii) A line segment between the points corresponding to any two elements $x$ and $y$ of the poset is included in the drawing if and only if $x$ covers $y$, or, $y$ covers $x$.
Consider the set $X = \{1, 2, 3\}$. Let the power set of $X$ be $\rho(X)$. Then

# 3. Fibonacci numbers

Q.3.1 Define Fibonacci series. Find 10-th Fibonacci number.

Answer: The Fibonacci series has been defined by the following recurrence relation: $F_{n+1} = F_n + F_{n-1}$ with base conditions: $F_1 = 1, F_2 = 1$. Based on the above recurrence relation and base conditions, we compute the following numbers.

$F_3 = F_2 + F_1 = 1 + 1 = 2$
$F_4 = F_3 + F_2 = 2 + 1 = 3$
$F_5 = F_4 + F_3 = 3 + 2 = 5$
$F_6 = F_5 + F_4 = 5 + 3 = 8$
$F_7 = F_6 + F_5 = 8 + 5 = 13$
$F_8 = F_7 + F_6 = 13 + 8 = 21$
$F_9 = F_8 + F_7 = 21 + 12 = 34$
$F_{10} = F_9 + F_8 = 34 + 21 = 55$

Q.3.2 The Fibonacci numbers can be extended to zero and negative indices using the relation $F_n = F_{n+2} - F_{n+1}$. Calculate Fibonacci numbers from $F_0$ to $F_{-5}$. Find a general formula for $F_{-n}$ in terms of $F_n$. Prove your result.

Answer: Fibonacci numbers from $F_0$ to $F_{-5}$ are given below:

$F_0 = F_2 - F_1 = 0$
$F_{-1} = F_1 - F_0 = 1 - 0 = 1$
$F_{-2} = F_0 - F_1 = 0 - 1 = -1$
$F_{-3} = F_{-1} - F_{-2} = 1 - (-1) = 2$
$F_{-4} = F_{-2} - F_{-3} = -1 - 2 = -3$
$F_{-5} = F_{-3} - F_{-4} = 2 - (-3) = 5$

It can be shown that $F_{-n} = (-1)^{n+1} F_n$. We shall prove the formula using mathematical induction on $n$.

Based on the above calculations, $F_{-1} = 1 = (-1)^{1+1} F_1$.

$F_{-2} = -1 = (-1)^{2+1} F_2$ [Q.3.1]

So, the formula holds true for $n = 1$ and $n = 2$.

We assume that the formula is true for $n \leq k$. We shall prove that the formula is true for $n = k + 1$.

Now, $F_{-(k+1)} = F_{-(k+1)+2} - F_{-(k+1)+1}$ [From definition]

$= F_{-(k-1)} - F_{-k} = (-1)^k F_{k-1} - (-1)^{k+1} F_k$ [Induction hypothesis]

$= (-1)^{k+2}(F_{k-1} + F_k)$

$= (-1)^{(k+1)+1} F_{k+1}$ [From recurrence relation]

Q.3.3 Obtain the generating function of Fibonacci numbers.

Answer: The generating function is given by $f(x) = \sum_{n=1}^{\infty} F_n x^n$.

Then $f(x) = F_1 x + F_2 x^2 + F_3 x^3 + F_4 x^4 + \ldots$

Also, $x f(x) = F_1 x^2 + F_2 x^3 + F_3 x^4 + F_4 x^5 + \ldots$

Also, $x^2 f(x) = F_1 x^3 + F_2 x^4 + F_3 x^5 + F_4 x^6 + \ldots$

Now, $(1 - x - x^2) f(x) = f(x) - x f(x) - x^2 f(x)$

We apply the facts that $F_1 = F_2 = 1$ and $F_{n+1} - F_n - F_{n-1} = 0$.

Then $f(x) - x f(x) - x^2 f(x) = F_1 x + x^2(F_2 - F_1) + x^3(F_3 - F_2 - F_1) + \ldots$.

or, $f(x) - x f(x) - x^2 f(x) = F_1 x = x$

Thus, $(1 - x - x^2) f(x) = x$

or, $f(x) = \frac{x}{1 - x - x^2}$.

Q.3.4 State some properties of Fibonacci numbers.

Answer: Some interesting properties and proofs of Fibonacci numbers are stated below:

*Property* 1: $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$ $\hfill$ (3.1)

Proof: $F_1 = F_3 - F_2$

$F_2 = F_4 - F_3$

$F_3 = F_5 - F_4$

$\ldots$

$F_{n-1} = F_{n+1} - F_n$

$F_n = F_{n+2} - F_{n+1}$

By adding the above $n$ statements, $\sum_{i=1}^{n} F_i = F_{n+2} - F_2$

$= F_{n+2} - 1 \; [F_2 = 1]$.

*Property* 2: $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$ $\hfill$ (3.2)

Proof: $F_1 = F_2$

$F_3 = F_4 - F_2$

$F_5 = F_6 - F_4$

$\ldots$

$F_{2n-1} = F_{2n} - F_{2n-2}$

By adding above $2n - 1$ statements, $\sum_{i=1}^{n} F_{2i-1} = F_{2n}$

*Property* 3: $F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} - 1$ $\hfill$ (3.3)

Proof: From (3.1), $\sum_{i=1}^{2n} F_i = F_{2n+2} - 1$ $\hfill$ (3.4)

$(3.4) - (3.2) \Rightarrow$

$\sum_{i=1}^{n} F_{2i} = F_{2n+2} - 1 - F_{2n}$

$= F_{2n+2} - F_{2n} - 1$

$= F_{2n+1} - 1$

*Corollary* 3.1: $F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-1} - F_{2n}$
$= -F_{2n-1} + 1$ (3.5)
Proof: From (3.2) and (3.3), $F_1 - F_2 + F_3 - F_4 + \cdots + F_{2n-1} - F_{2n}$
$= F_{2n} - (F_{2n+1} - 1)$
$= F_{2n} - F_{2n+1} + 1$
$= -F_{2n-1} + 1$

*Corollary* 3.2: $F_1 - F_2 + F_3 - F_4 + \cdots - F_{2n} + F_{2n+1} = F_{2n} + 1$ (3.6)
Proof: By adding $F_{2n+1}$ to both sides of (3.5), the result follows.

*Corollary* 3.3: $F_1 - F_2 + F_3 - F_4 + \ldots (-1)^{n+1} F_n$
$= (-1)^{n+1} F_{n-1} + 1$ (3.7)
Proof: Refer to result (3.5). $2n$ is an even number. Let $2n = k$.
From (3.5), $F_1 - F_2 + F_3 - F_4 + \cdots + F_{k-1} - F_k = (-1)^{k+1} F_{k-1} + 1$ (3.8)
Refer to result (3.6). $2n + 1$ is an odd number. Let $2n + 1 = k$.
From (3.6), $F_1 - F_2 + F_3 - F_4 + \cdots - F_{k-1} + F_k = (-1)^{k+1} F_{k-1} + 1$ (3.9)
Combining (3.8) and (3.9), the Corollary 3.3 follows.

*Property* 4: $F_1^2 + F_2^2 + F_3^2 + \cdots + F_n^2 = F_n F_{n+1}$
Proof: We note that $F_k F_{k+1} - F_{k-1} F_k = F_k (F_{k+1} - F_{k-1}) = F_k^2$
$F_1^2 = F_1 F_2$ (as $F_1 = F_2 = 1$)
$F_2^2 = F_2 F_3 - F_1 F_2$
$F_3^2 = F_3 F_4 - F_2 F_3$
$\cdots$
$F_n^2 = F_n F_{n+1} - F_{n-1} F_n$
By adding the preceeding statements, Property 4 follows.

Q.3.5 Solve the following recurrence relation.
$u_n = u_{n-1} + u_{n-2}$ (3.10)
where, $u_1 = u_2 = 1$
Answer: Using (3.10), $\sum_{n=3}^{\infty} u_n x^n = \sum_{n=3}^{\infty} u_{n-1} x^n + \sum_{n=3}^{\infty} u_{n-2} x^n$
Let $G(x) = u_1 + u_2 x + u_3 x^2 + \cdots + u_{n-1} x^{n-2} + u_n x^{n-1} + \ldots$
Therefore, $x \sum_{n=3}^{\infty} u_n x^{n-1} = x^2 \sum_{n=3}^{\infty} a_{n-1} x^{n-2} + x^3 \sum_{n=3}^{\infty} a_{n-2} x^{n-3}$
or, $G(x) - u_1 - u_2 x = x[G(x) - u_1] + x^2 G(x)$, for $x \neq 0$
or, $G(x)[1 - x - x^2] = u_1 + u_2 x - u_1 x = u_1 = 1$, since $u_1 = u_2 = 1$
or, $G(x) = (1 - x - x^2)^{-1} = \frac{1}{1-x-x^2}$
or, $G(x) = \frac{1}{(x-\alpha)(x-\beta)}$, where $\alpha, \beta$ are the roots of the following equation:
$1 - x - x^2 = 0$
or, $x^2 + x - 1 = 0$